

# Lecture 15 - Current

## A Puzzle...

Suppose an infinite grounded conducting plane lies at  $z = 0$ . A charge  $q$  is located at a height  $h$  above the conducting plane. Show in three different ways that the potential below the conducting plane is zero everywhere.

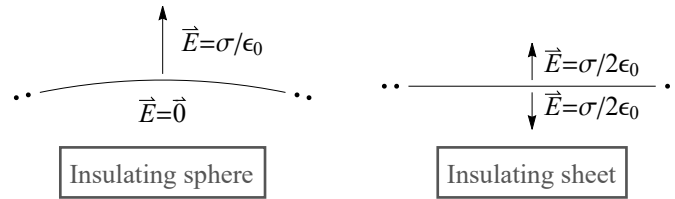
### Solution

1. We can consider the space  $z < 0$  to be inside of a giant hemispherical conductor which is bounded by  $z = 0$  from above and by the infinite hemisphere everywhere below and around it. The potential on this entire conductor is 0. One obvious solution is that the potential everywhere in this region is zero, and the Uniqueness Theorem guarantees that this is the only solution.
2. We could use an image charge to solve this problem. Recall that image charges are always located outside of regions(s) where the potential is to be calculated. Therefore, we are free to use any image charges we want in the region  $z > 0$ . One wise choice would be to put a point charge  $-q$  right on top of the charge  $+q$ , which would cancel all of the charges and yield potential zero everywhere. This result (which is only valid for  $z < 0$ ), tells us that the potential below the conducting plate is zero everywhere.  $\square$
3. In your book (more precisely, Figure 3.12), the case of a finite (ungrounded) circular disk conductor was discussed. In such a case, a total charge  $+\frac{q}{2}$  distributes itself (roughly) evenly on the underside of the conducting sheet. In the limit as the radius  $R$  of the disk goes to infinity, we recoup the infinite conducting plane, which would have charge density zero. Although this sheet is not grounded, it clearly has potential zero, and therefore grounding it will not change the charge distribution. Therefore, there will be zero charge density, and therefore zero potential, everywhere for  $z < 0$ .
4. Imagine that the infinite conducting plane is actually a conducting spherical shell with radius  $R$  and center  $(0, 0, -R)$  in Cartesian coordinates. The limit  $R \rightarrow \infty$  would yield the same geometry as the conducting plane, and the electric potential inside of the grounded spherical cavity would be zero everywhere (this is the unique solution to Laplace's equation with  $\phi = 0$  everywhere on the boundary). Alternatively, you can imagine a sphere with radius  $R$  and center  $(0, 0, R)$  which contains the point charge. In this case, there would be a charge  $\frac{q}{4\pi R^2}$  distribution uniformly across the surface of the sphere, which goes to zero as  $R \rightarrow \infty$ , and therefore leads to zero potential everywhere outside the sphere (which corresponds to  $z < 0$  in the case  $R \rightarrow \infty$ ).

## A Word of Warning

Solutions #3 and #4 probably disturbed you quite a bit, since we have never used the limiting case of an infinite sphere to characterize an infinite plane before this. And indeed, we could not have because the parallels between these two geometries only hold for conductors and not insulators.

To see this, consider a large insulating spherical shell with radius  $R$  and constant charge density  $\sigma$ . The electric field inside the sphere is  $\vec{E} = \vec{0}$  everywhere while the electric field outside is  $\vec{E} = \frac{k(4\pi R^2 \sigma)}{r^2} \hat{r} = \frac{\sigma}{\epsilon_0} \frac{R^2}{r^2} \hat{r}$ . As we approach infinitesimally close to the sphere  $r \rightarrow R$ , we obtain the result  $\vec{E} = \frac{\sigma}{\epsilon_0} \hat{r}$ . Place the center of the sphere at  $(0, 0, -R)$  and take the limit  $R \rightarrow \infty$ . Although locally, the geometry around the origin resembles an infinite sheet, the global geometry certainly matters. Indeed, if we considered a real insulating infinite sheet on the  $x$ - $y$  plane, the electric field would be  $\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{z}$  above the sheet and  $\vec{E} = -\frac{\sigma}{2\epsilon_0} \hat{z}$  below the sheet.



However, if we now consider an infinitely large conducting sphere and an infinite conducting sheet, the situation is different. In this case, we have the Uniqueness Theorem which tells us that if we have specified the potential on all conducting boundaries (and at infinity), then the electric potential has a unique solution. In this case, the specific geometry you choose does not matter, provided that you do not change the boundary conditions. In the puzzle above, the boundary conditions are  $V = 0$  everywhere along the  $x$ - $y$  plane, together with  $V \rightarrow 0$  at infinity. Both the infinite grounded sphere and the infinite grounded sheet satisfy these conditions, and hence by the Uniqueness Theorem they must yield the same solution, as indeed they do.

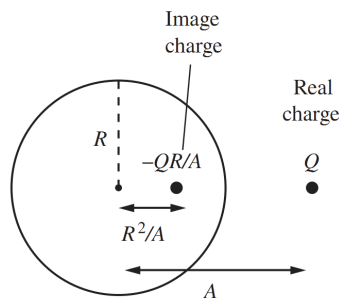
Similarly, as we will show below, the electric force felt by a charge  $Q$  as it approaches the surface of a grounded conducting sphere equals the same force felt by a charge  $Q$  as it approaches the surface of a grounded infinite sheet.

## Advanced Section: Image Charge for Spheres

### Image Charge for a Grounded Spherical Shell

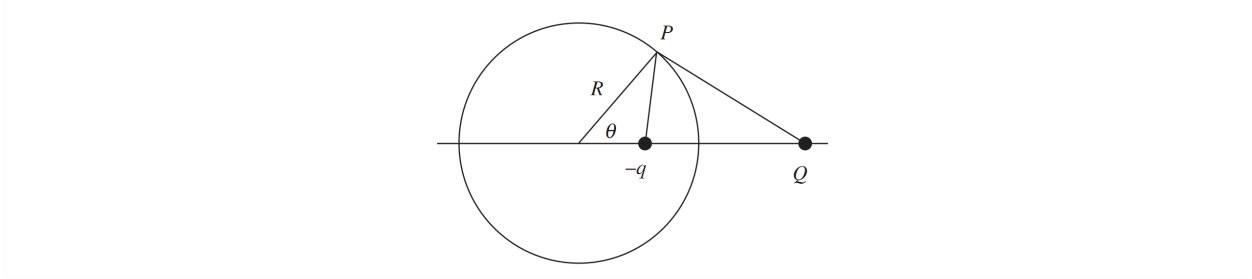
#### Example

A charge  $Q$  lies at a distance  $A$  outside of a grounded conducting sphere of radius  $R$ . Prove that placing an image charge  $-q = -\frac{QR}{A}$  at a distance  $\frac{R^2}{A}$  from the center of the sphere (along the same line as the charge  $Q$ ) yields a constant potential 0 everywhere on the sphere. Use this to determine the force felt from the charge  $Q$  from the conducting sphere.



#### Solution

Let us determine the potential at point  $P$ , which lies at an angle  $\theta$  relative to the line connecting the center of the sphere to the image charges.



The distance from the charge  $Q$  to the point  $P$  equals

$$r_Q = (R^2 + A^2 - 2RA \cos[\theta])^{1/2} \quad (1)$$

while the distance from  $-q$  to  $P$  equals

$$r_{-q} = \left( R^2 + \left( \frac{R^2}{A} \right)^2 - 2 \frac{R^3}{A} \cos[\theta] \right)^{1/2} \quad (2)$$

Therefore, the potential at point  $P$  equals

$$\frac{kQ}{r_Q} + \frac{k(-q)}{r_{-q}} = \frac{kQ}{(R^2 + A^2 - 2RA \cos[\theta])^{1/2}} - \frac{k \frac{QR}{A}}{\left( R^2 + \left( \frac{R^2}{A} \right)^2 - 2 \frac{R^3}{A} \cos[\theta] \right)^{1/2}} = 0 \quad (3)$$

Since  $\theta$  was arbitrary, this shows that the potential is 0 across the entire sphere. What an incredible result!

With this in hand, we can now easily calculate the force on the charge  $Q$  since it equals the force that the image charge  $-q$  would exert upon  $Q$  in the diagram above, which equals

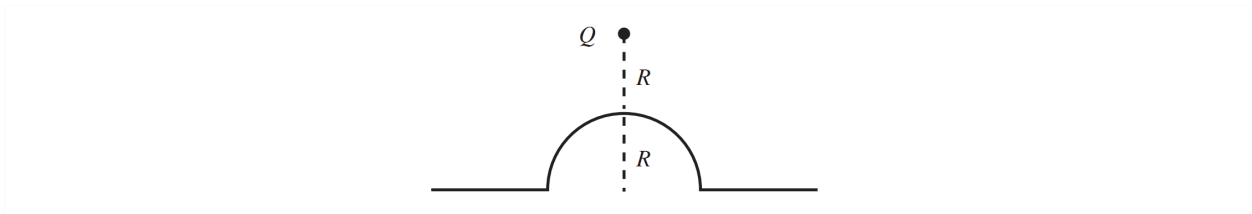
$$\begin{aligned} \vec{F} &= \frac{k(-q)Q}{A - \frac{R^2}{A}} \hat{x} \\ &= - \frac{k \left( \frac{QR}{A} \right) Q}{A - \frac{R^2}{A}} \hat{x} \\ &= - \frac{k R Q^2}{A^2 - R^2} \hat{x} \end{aligned} \quad (4)$$

as desired. There are many exciting limits to check. For example, Exercise 3.46 examines the limit when the real charge is very close to the grounded sphere, in which case the setup must reduce to the image charge problem of a grounded infinite plane.  $\square$

## Bump on a Plane

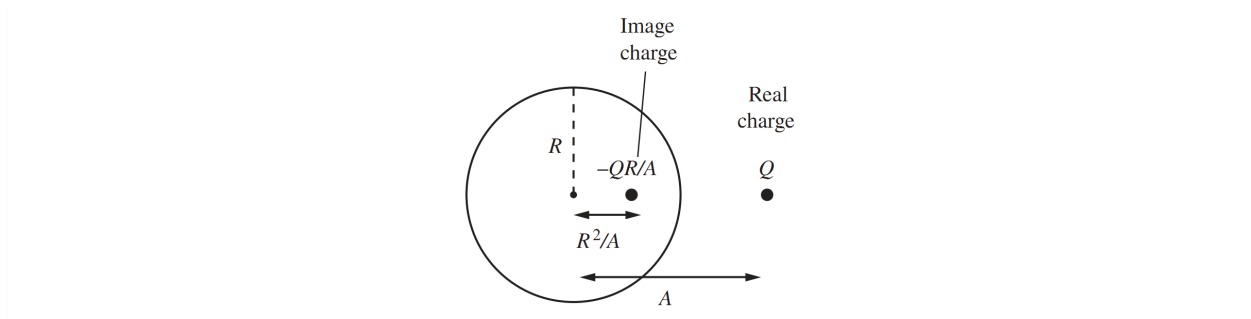
### Example

An infinite conducting plane has a hemispherical bump on it with radius  $R$ . A point charge  $Q$  is located a distance  $R$  above the top of the hemisphere. Using the result of the previous problem, find the image charges needed to make the electric field perpendicular to the plane and the hemisphere at all points.



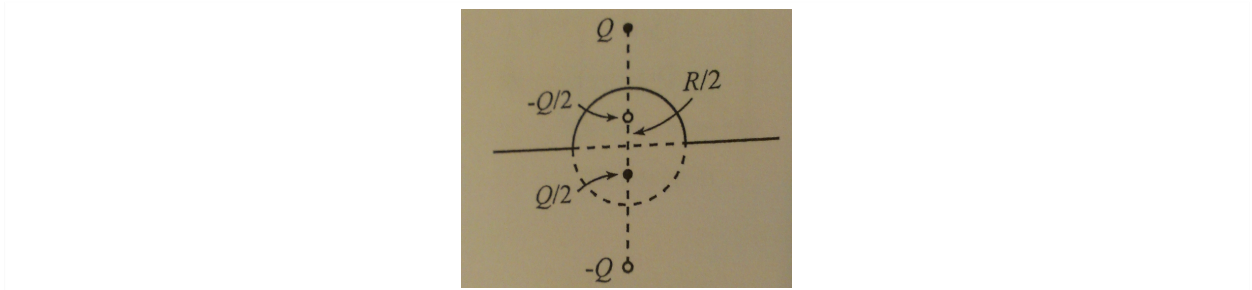
### Solution

Recalling the result of the previous diagram, where  $A = 2R$ ,



we can try putting an image charge  $-\frac{QR}{A} = -\frac{Q}{2}$  at a distance  $\frac{R^2}{A} = \frac{R}{2}$  above the center of the hemispherical bump.

But that will not be enough to ensure that the plane is equipotential; in order to do that we need to add two identical but opposite charges below the plane.



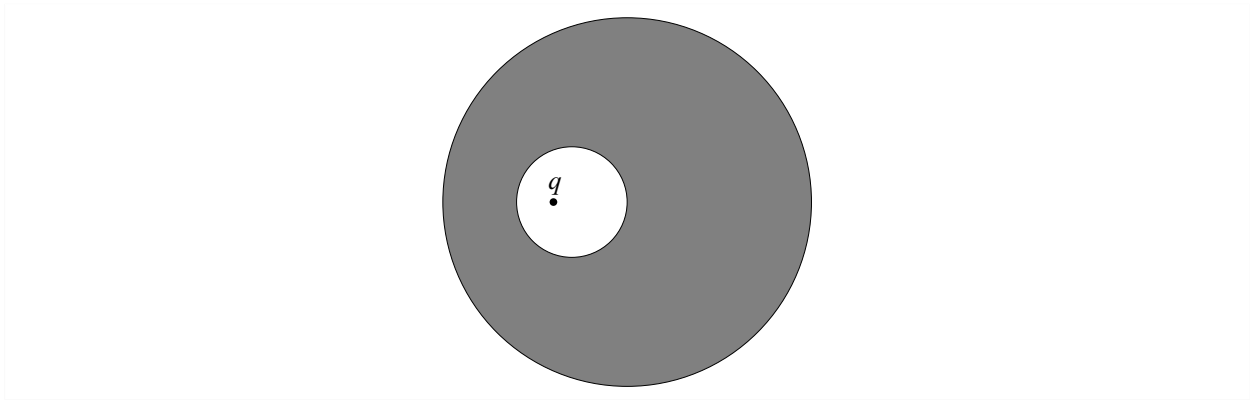
But wait, don't the two extra charges that we just inserted mess up the potential on the hemisphere...?

No, they don't! Because by the results of the previous problem, the bottom two charges (as well as the top two charges) cause 0 potential all over the sphere. Furthermore, the inner two charges and the outer two charges cause 0 potential all over the plane. So in fact, we have created an equipotential surface consisting of the union of the plane and the sphere (i.e. the solid and dashed surfaces in the diagram above), even though we don't care about the bottom of the hemisphere or the equatorial disk inside the sphere (i.e. the dashed lines).

This solution can be easily generalized so that the charge  $Q$  is any distance above the hemispherical bump (but it won't work for a charge  $Q$  inside of the hemispherical bump, since you could not setup image charges only in the outside region above the bump). A more sophisticated (and probably numerical) method would be required to compute the potential in such a case.  $\square$

## Charge Inside a Spherical Cavity

Consider a neutral spherical conductor with an off-center spherical cavity.



Let's summarize what we know about this situation:

- By Gauss's Law, a total charge  $-q$  must be induced on the inner surface of the conductor, and by neutrality a charge  $q$  must be uniformly distributed over the outer surface of the conductor. Therefore, the conductor will be at the potential  $\frac{kq}{R}$  where  $R$  is the radius of the conductor (ignoring the cavity), regardless of the position of the charge  $q$
- If the point charge  $q$  is placed at the center of the cavity, then the charge  $-q$  will be uniformly distributed over the conductor's inner surface, and the force on the point charge  $q$  will be zero by symmetry
- If the point charge  $q$  is not in the center, will it feel a force? Absolutely. We know that if the total charge  $-q$  on the inside of the conductor was spread uniformly, then the point charge  $q$  would feel no force anywhere in the cavity. But as the point charge  $q$  moves away from the center, the charge  $-q$  will accumulate more on the part of the cavity closer to  $q$ , which will lead to a force on  $q$
- How large will the force on  $q$  be? We can use the method of images to find an exact force. Recall that conductor problems are equivalent if the boundary conditions are the same. In this case, we know exactly what the electric field will be outside the conductor and within the meat of the conductor. All that remains is to find the electric field inside the conductor's cavity. This last problem is equivalent to finding the force on a point charge  $q$  inside of a conducting sphere of radius  $R$  (the same radius as that of the conductor's cavity) and whose potential is  $\frac{kq}{R}$ .
  - Define  $A$  to be the distance between the point charge  $q$  and the center of the sphere. We can use the image charge configuration by using an image charge  $-\frac{qA}{R}$  located at a distance  $\frac{R^2}{A}$  from the center of the sphere
  - This image configuration will ensure that the potential is zero everywhere across the spherical conductor. Since we want the potential of the conductor to be  $\frac{kq}{R}$ , we add a charge  $q$  spread uniformly across the outer surface of the sphere. This uniformly spread image charge (which we note is on the outside of the region of interest and hence valid) will only ensure that the potential of the sphere is correct but will not actually change the force felt by  $q$
  - As a result, the force felt by the point charge when it is at a distance  $A$  from the center of the spherical cavity is  $\frac{\frac{A}{R} q^2}{\left(A - \frac{R^2}{A}\right)^2}$  pointing away from the center of the sphere

## Theory

### Current

Current is measured in amperes ( $1 A = 1 \frac{C}{s}$ ). The *current density*  $\vec{J}$  measures current per unit area,

$$\vec{J} = n q \vec{u} \quad (5)$$

where  $n$  is the density,  $q$  is the charge, and  $\vec{u}$  is the velocity of the moving charge. The current flowing along a surface  $S$  is given by

$$I = \int_S \vec{J} \cdot d\vec{a} \quad (6)$$

For a time-independent current distribution, the charge going into any surface must equal the charge flowing out of the surface (if the charge is accumulating, the situation is not time-independent). In such a case, we can write

$$\vec{\nabla} \cdot \vec{J} = 0 \quad (\text{steady state current flow}) \quad (7)$$

### Circuits

Circuits can be analyzed using the following three rules (known as Kirchoff's rules):

- 1. Ohm's Law:** The current through each element must equal the voltage across that element divided by the resistance of the element.
- 2. Charge conservation:** The sum of current going into a junction equals the sum of current going out of that junction. This is charge conservation,  $\vec{\nabla} \cdot \vec{J} = 0$ , in circuit language.
- 3. Closed loops in electrostatic fields:** The sum of the potential differences taken in order around a loop of the network, a path beginning and ending at the same node, is zero. This is  $\oint \vec{E} \cdot d\vec{s} = 0$  for a static electric field in circuit language.

## Current Problems

### Synchrotron Current

#### Example

In a 6 gigaelectron-volt ( $1 \text{ GeV} = 10^9 \text{ eV}$ ) electron synchrotron, electrons travel around the machine in an approximately circular path 240 meters long. It is normal to have about  $10^{11}$  electrons circling on this path during a cycle of acceleration. The speed of the electrons is practically that of light. What is the current?

#### Solution

The number of round trips that each electron makes per second is  $\frac{3 \times 10^8 \frac{m}{s}}{240 m} = 1.25 \times 10^6 \frac{1}{s}$ , so that the charge passing any given point equals

$$(1.25 \times 10^6 \frac{1}{s}) N e = (1.25 \times 10^6 \frac{1}{s}) (10^{11}) (1.6 \times 10^{-19} C) = 0.02 A \quad (8)$$

The point of this simple problem is to demonstrate that nothing in our definition of current as rate of transport requires the velocities of the charge carriers to be non-relativistic and that there is no rule against a given charged particle getting counted many times during a second as part of the current.  $\square$

### Combining the Current Densities

Example

We have  $5 \times 10^{16}$  doubly charged positive ions per  $m^3$ , all moving west with a speed of  $10^5 \frac{m}{s}$ . In the same region there are  $10^{17}$  electrons per  $m^3$  moving northeast with a speed of  $10^6 \frac{m}{s}$ . What are the magnitude and direction of  $J$ ?

Solution

Current density is given by  $\vec{J} = q n \vec{u}$ , so the magnitude of the two  $\vec{J}$ 's are

$$J_{\text{ion}} = (2e) n_{\text{ion}} u_{\text{ion}} = 2 (1.6 \times 10^{-19} \text{ C}) (5 \times 10^{16} \frac{1}{m^3}) (10^5 \frac{m}{s}) = 1.6 \times 10^3 \frac{A}{m^2} \quad (9)$$

$$J_{\text{elec}} = (e) n_{\text{elec}} u_{\text{elec}} = (1.6 \times 10^{-19} \text{ C}) (10^{17} \frac{1}{m^3}) (10^6 \frac{m}{s}) = 1.6 \times 10^4 \frac{A}{m^2} \quad (10)$$

$\vec{J}_{\text{ion}}$  points west and  $\vec{J}_{\text{elec}}$  points southwest (due to the negative charge of the electron). Therefore, the components of the total current density equals

$$\begin{aligned} \vec{J}_{\text{tot}} &= \vec{J}_{\text{ion}} + \vec{J}_{\text{elec}} \\ &= \left\langle -J_{\text{ion}} - \frac{J_{\text{elec}}}{\sqrt{2}}, -\frac{J_{\text{elec}}}{\sqrt{2}} \right\rangle \\ &= \left\langle -1.29 \times 10^4, -1.13 \times 10^4 \right\rangle \end{aligned} \quad (11)$$

The flipped direction of  $\vec{J}_{\text{elec}}$  is easy to remember if you stick in the actual negative charge of the electron in the current density equation,  $\vec{J}_{\text{elec}} = (-e) n_{\text{elec}} \vec{u}_{\text{elec}}$ .  $\square$

## Resistors

Example

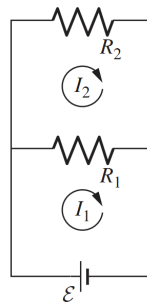
What is the equivalent resistance for adding two resistors (with resistance  $R_1$  and  $R_2$ ) in series.

Solution

In a series circuit, the current through each of the components is the same, and the voltage across the circuit is the sum of the voltages across each component. For a current  $I$ , the voltage drop across each individual resistor equals  $I R_1$  and  $I R_2$ , and therefore the total voltage drop is  $I (R_1 + R_2)$ . Therefore, these two resistors are identical to a single resistor with resistance  $R = R_1 + R_2$ .  $\square$

Example

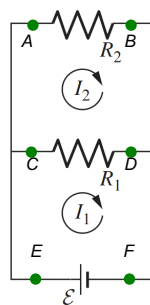
By solving the loop equations for the circuit shown below, derive the rule for adding resistors in parallel.

Solution

To obey Kirchhoff's second rule (current in equals current out), we define the currents  $I_1$  and  $I_2$  to flow along the loops shown. The current through the middle segment equals  $I_1 - I_2$  to the right. If we ultimately end up with a negative value, it means that we chose the wrong direction for one of our currents, but that is perfectly fine (just like finding a negative force actually means that the force points in the opposite direction).

We now want to use Kirchhoff's third rule, which states the voltage drop across any loop is 0. To do this, we label different points on our circuit as  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ , and  $F$ , as shown below. We now pick a direction (we will choose

clockwise for each loop) and calculate the total voltage drop around each loop.



For the top loop starting at point  $A$  we move clockwise to point  $B$ . The voltage drop (i.e. the difference between the voltage  $V_B$  at point  $B$  and the voltage  $V_A$  at point  $A$ ) is  $-I_2 R_2$  across this resistor. We then continue clockwise until point  $D$  (where the voltage is the same as point  $B$ ) and then to point  $C$ . The voltage drop from  $D$  to  $C$  due to the current  $I_2$  is  $-I_2 R_1$ , while the voltage drop due to the current  $I_1$  is  $+I_1 R_1$  (note the plus sign! For resistors there is a voltage drop *in the direction of the current*, so for  $I_1$  the voltage at  $D$  is lower than the voltage at  $C$ , but since we are looking at the top loop we are moving in the reverse direction and computing the voltage at  $C$  minus the voltage at  $D$ ). Therefore, Kirchhoff's third rule for the top loop is

$$-(I_2 - I_1) R_1 - I_2 R_2 = 0 \quad (12)$$

Similarly, moving clockwise around the bottom loop, the voltage drop from  $C$  to  $D$  is  $(I_2 - I_1) R_1$  (i.e. the reverse of the voltage drop we just computed from  $D$  to  $C$ ) plus the voltage drop in moving from  $F$  to  $E$ . Since we are current is flowing from the negative plate to the positive plate in the battery (the positive plate always is marked by the larger bar), the voltage drop here is the positive value  $\mathcal{E}$ . Therefore, Kirchhoff's third rule for the bottom loop is

$$\mathcal{E} + (I_2 - I_1) R_1 = 0 \quad (13)$$

Note that adding these two loop equations yields  $\mathcal{E} - I_2 R_2 = 0$ , which is the loop equation around the whole circuit. Thus we can solve for  $I_1$ ,

$$I_1 = \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \mathcal{E} \equiv \frac{\mathcal{E}}{R_{\text{eff}}} \quad (14)$$

where  $\frac{1}{R_{\text{eff}}} = \frac{1}{R_1} + \frac{1}{R_2}$ . The current through the battery is  $I_1$ , so  $\mathcal{E} = I_1 R_{\text{eff}}$  tells us that the effective resistance is  $R_{\text{eff}}$ .

□

Once you know how to add a circuit element (such as a resistor) in series and parallel, you can simplify any circuit with these elements down. For example, three resistors  $R_1$ ,  $R_2$ ,  $R_3$  in parallel would have resistance

$$\frac{1}{R_{\text{eff}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \quad (15)$$

or equivalently

$$R_{\text{eff}} = \frac{R_1 R_2 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1} \quad (16)$$

Some wise words from [Wikipedia](#):

As an example, consider a very simple circuit consisting of four light bulbs and one 6 V battery. If a wire joins the battery to one bulb, to the next bulb, to the next bulb, to the next bulb, then back to the battery, in one continuous loop, the bulbs are said to be in series. If each bulb is wired to the battery in a separate loop, the bulbs are said to be in parallel.

If the four light bulbs are connected in series, there is same current through all of them, and the voltage drop is 1.5 V across each bulb, which may not be sufficient to make them glow. If the light bulbs are connected in parallel, the currents through the light bulbs combine to form the current in the battery, while the voltage drop is 6 V



across each bulb and they all glow.

In a series circuit, every device must function for the circuit to be complete. One bulb burning out in a series circuit breaks the circuit. In parallel circuits, each light has its own circuit, so all but one light could be burned out, and the last one will still function.

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## Mathematica Initialization